

Lorentz transformations and the wave equation

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Abstract

In this note we explicitly show how the Lorentz transformations can be derived by demanding form invariance of the d'Alembert operator in inertial reference frames.

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1. Introduction

The homogeneous scalar wave equation is form invariant under the Lorentz transformations¹. It is therefore reasonable to expect that these transformations may be derived by requiring form invariance of the homogeneous scalar wave equation. Although this expectation turns out to be correct, its explicit demonstration does not seem to be widely known². In this note we derive the Lorentz transformations in their standard configuration by demanding form invariance of the d'Alembert operator in inertial reference frames [5]. This relatively simple derivation of the Lorentz transformations could be suitable for undergraduate courses of electromagnetism and special relativity.

¹ It is well-known that the wave equation is invariant under the Lorentz transformations but the explicit demonstration of this statement is not usually presented in standard textbooks. Some popular undergraduate textbooks like [1] and also some known graduate textbooks like [2, 3] explicitly show that the homogeneous wave equation is not form invariant under the Galilean transformations but they do not explicitly demonstrate the form invariance of this equation under the Lorentz transformations. In the book [4], we find the explicit demonstration of the invariance of the wave equation under the Lorentz transformations.

² At first sight, the derivation of the Lorentz transformations from the form invariance of the wave equation seems to be a natural task. However, the present author has not been able to find some textbook in which such a derivation is presented.

2. Lorentz transformations from the invariance of the d'Alembertian

Consider the standard configuration in which two inertial frames S and S' are in relative motion with speed v along their common xx' direction. The origins of the two frames coincide at the instant $t = t' = 0$. The coordinates transverse to the relative motion of the frames S and S' are assumed to be invariant: $y' = y$ and $z' = z$. The corresponding derivative operators are also assumed to be invariant: $\partial/\partial y = \partial/\partial y'$ and $\partial/\partial z = \partial/\partial z'$ which imply $\partial^2/\partial y^2 = \partial^2/\partial y'^2$ and $\partial^2/\partial z^2 = \partial^2/\partial z'^2$. Our problem reduces then to find the transformation rules for $\partial^2/\partial x^2$ and $\partial^2/\partial t^2$. The invariance of the reduced d'Alembert operator can be expressed as

$$\frac{\partial^2}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} = \frac{\partial^2}{\partial x'^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t'^2}. \quad (1)$$

We observe that in stating this invariance, we are assuming the postulate of relativity, which states that the laws of physics are the same in all inertial frames and the postulate of the constancy of the speed of light, which states that the speed of light in vacuum has the same value in all inertial frames. Equation (1) can be factored as follows

$$\left(\frac{\partial}{\partial x} - \frac{1}{c} \frac{\partial}{\partial t} \right) \left(\frac{\partial}{\partial x} + \frac{1}{c} \frac{\partial}{\partial t} \right) = \left(\frac{\partial}{\partial x'} - \frac{1}{c} \frac{\partial}{\partial t'} \right) \left(\frac{\partial}{\partial x'} + \frac{1}{c} \frac{\partial}{\partial t'} \right). \quad (2)$$

By assuming linearity for the involved transformations of operators, we can write

$$\left(\frac{\partial}{\partial x} - \frac{1}{c} \frac{\partial}{\partial t} \right) = A \left(\frac{\partial}{\partial x'} - \frac{1}{c} \frac{\partial}{\partial t'} \right), \quad (3)$$

$$\left(\frac{\partial}{\partial x} + \frac{1}{c} \frac{\partial}{\partial t} \right) = A^{-1} \left(\frac{\partial}{\partial x'} + \frac{1}{c} \frac{\partial}{\partial t'} \right), \quad (4)$$

where the factor A is independent of the derivative operators but can depend on the velocity v . In order to determine A , we demand that the expected linear transformation relating primed and unprimed time-derivative operators should appropriately reduce to the corresponding Galilean transformation³: $\partial/\partial t = \partial/\partial t' - v\partial/\partial x'$. Our demand is consistent with a linear transformation of the general form: $\partial/\partial t = F(v)(\partial/\partial t' - v\partial/\partial x')$, where $F(v)$ depends on the velocity v so that $F(v) \rightarrow 1$ when $v \ll c$. From this general transformation it follows that if $\partial/\partial t = 0$ then $\partial/\partial t' = v\partial/\partial x'$ because $F(v) \neq 0$. Using this result in equations (3) and (4) we obtain

$$\frac{\partial}{\partial x} = A \left(\frac{\partial}{\partial x'} - \frac{v}{c} \frac{\partial}{\partial x'} \right), \quad (5)$$

$$\frac{\partial}{\partial x} = A^{-1} \left(\frac{\partial}{\partial x'} + \frac{v}{c} \frac{\partial}{\partial x'} \right). \quad (6)$$

By combining these equations we get the expressions

$$A = \frac{(1 + v/c)^{1/2}}{(1 - v/c)^{1/2}} = \gamma(1 + v/c), \quad (7)$$

³ See, for example, [1, footnote 1, p 496].

$$A^{-1} = \frac{(1 - v/c)^{1/2}}{(1 + v/c)^{1/2}} = \gamma(1 - v/c), \quad (8)$$

where $\gamma = 1/\sqrt{1 - v^2/c^2}$. Using equations (7) and (8) into equations (3) and (4), we obtain the transformation laws⁴:

$$\left(\frac{\partial}{\partial x} - \frac{1}{c} \frac{\partial}{\partial t} \right) = \gamma \left(1 + \frac{v}{c} \right) \left(\frac{\partial}{\partial x'} - \frac{1}{c} \frac{\partial}{\partial t'} \right), \quad (9)$$

$$\left(\frac{\partial}{\partial x} + \frac{1}{c} \frac{\partial}{\partial t} \right) = \gamma \left(1 - \frac{v}{c} \right) \left(\frac{\partial}{\partial x'} + \frac{1}{c} \frac{\partial}{\partial t'} \right). \quad (10)$$

By adding and subtracting these equations, we obtain the transformation laws connecting unprimed and primed derivative operators

$$\frac{\partial}{\partial x} = \gamma \left(\frac{\partial}{\partial x'} - \frac{v}{c^2} \frac{\partial}{\partial t'} \right), \quad (11)$$

$$\frac{\partial}{\partial t} = \gamma \left(\frac{\partial}{\partial t'} - v \frac{\partial}{\partial x'} \right), \quad (12)$$

which must be completed with the transformations $\partial/\partial y = \partial/\partial y'$ and $\partial/\partial z = \partial/\partial z'$. These transformation laws are the Lorentz transformations for derivative operators of the standard configuration. We note that equations (11) and (12) imply coordinate transformations of the form: $x' = x'(x, t)$, $y' = y$, $z' = z$ and $t' = t'(x, t)$. To find the explicit form of these transformations we can use equations (11) and (12) to obtain

$$\frac{\partial x'}{\partial x} = \gamma, \quad \frac{\partial x'}{\partial t} = -\gamma v, \quad \frac{\partial t'}{\partial t} = \gamma, \quad \frac{\partial t'}{\partial x} = -\frac{\gamma v}{c^2}. \quad (13)$$

The first relation in equation (13) implies (I): $x' = \gamma x + f_1(t)$, where $f_1(t)$ can be determined (up to a constant) by differentiating (I) with respect to the time t and using the second relation in equation (13): $\partial x'/\partial t = df_1(t)/dt = -\gamma v$. This last equality implies (II): $f_1(t) = -\gamma vt + x_0$, where x_0 is a constant. From (I) and (II) we obtain

$$x' = \gamma(x - vt) + x_0. \quad (14)$$

The third relation in equation (13) implies (III): $t' = \gamma t + f_2(x)$, where $f_2(x)$ can be obtained (up to a constant) from differentiating (III) with respect to x and using the last relation in equation (13): $\partial t'/\partial x = df_2(x)/dx = -\gamma v/c^2$. This last equality implies (IV): $f_2(x) = -\gamma vx/c^2 + t_0$, where t_0 is a constant. From (III) and (IV) we obtain

$$t' = \gamma(t - vx/c^2) + t_0. \quad (15)$$

The origins of the frames S and S' coincide at the time $t = t' = 0$ and therefore we have $x_0 = 0$ and $t_0 = 0$. In this way we obtain the Lorentz transformations of the standard configuration:

⁴ If we define the derivative operators: $\partial_\xi \equiv \partial/\partial x - (1/c)\partial/\partial t$, $\partial_\eta \equiv \partial/\partial x + (1/c)\partial/\partial t$, $\partial_{\xi'} \equiv \partial/\partial x' - (1/c)\partial/\partial t'$ and $\partial_{\eta'} \equiv \partial/\partial x' + (1/c)\partial/\partial t'$ then the set formed by the transformations in equations (9) and (10) together with the transformations $\partial/\partial y = \partial/\partial y'$ and $\partial/\partial z = \partial/\partial z'$ can compactly be expressed in the tensor form $\partial_A = D_A^B \partial_{B'}$, where $\partial_A = [\partial_\xi, \partial_\eta, \partial_y, \partial_z]$ and $\partial_{B'} = [\partial_{\xi'}, \partial_{\eta'}, \partial_{y'}, \partial_{z'}]$ (summation on the repeated index B is understood). The non-vanishing elements of the transformation matrix D_A^B are given by $D_1^1 = A$, $D_2^2 = A^{-1}$, $D_3^3 = 1$ and $D_4^4 = 1$ with $A = \gamma(1 + v/c)$ and $A^{-1} = \gamma(1 - v/c)$. This means that the matrix D is diagonal and therefore we can say, following Parker and Schmieg in [5], that $\partial_A = D_A^B \partial_{B'}$ represents a diagonal form of the Lorentz transformations (standard configuration) for the specified derivative operators.

$$x' = \gamma(x - vt), \quad t' = \gamma(t - vx/c^2), \quad (16)$$

which must be completed with the remaining transformations: $y' = y$ and $z' = z$.

3. Conclusion

There is no doubt that the wave equation is one of the most important equations in physics. This equation describes a great variety of physical phenomena. Therefore the symmetries associated with the wave equation are equally important. Perhaps the most famous of these symmetries is the Lorentz symmetry. The d'Alembert operator, the basic ingredient of the wave equation, is shown to be form invariant under the Lorentz transformations. In this note we have traveled the inverse route and demanded form invariance of the d'Alembert operator to obtain the Lorentz transformations in their standard configuration. This derivation relies on the postulates of special relativity; it enriches the list of the many derivations of the Lorentz transformations presented over the years [6].

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