Reply

Reply to Comments on 'Feynman's handwritten notes on electromagnetism and the idea of introducing potentials before fields'

José A Heras[1] and Ricardo Heras[2,*]

[1] Instituto de Geofísica, Universidad Nacional Autónoma de México, Ciudad de México 04510, Mexico
[2] Department of Physics and Astronomy, University College London, London WC1E 6BT, United Kingdom

E-mail: herasgomez@gmail.com and ricardo.heras.13@ucl.ac.uk

Abstract
We reply to some comments made by Davis (2020 *Eur. J. Phys.* **40** 018001) on our paper (2020 *Eur. J. Phys.* **41** 035202), by arguing that Davis's assertions are unsupported in some cases and are unsatisfactory in other cases.

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Davis [1] has made some comments on our recent paper [2] in which we discussed Feynman's idea of introducing potentials before fields in an alternate presentation of Maxwell's equations. We feel that a reply to Davis's comments is required, which could be useful for the readers of *European Journal of Physics*. To put in context our reply, let us recall that in 1963 Feynman sketched in some handwritten notes on an 'Alternate way to handle electrodynamics' the idea of introducing first the scalar and vector potentials before the electric and magnetic fields. These notes were recently discovered by Gottlieb [3]. De Luca *et al* [4, 5] as well as ourselves [2] have attempted to implement some of Feynman's ideas with the aim that they can be used in courses of electrodynamics. Davis [1] has joined the discussion by commenting on our paper [2] and therefore in this reply we are going to respond to Davis's comments.

1. In his comments [1], Davis first tried to interpret what he called the 'somewhat cryptic assertion made by Richard P Feynman' about the reality of the vector potential [6]: '**A** is as real as **B**—realer, whatever that means.' He then conjectured that in this assertion Feynman meant to say that '**A** is as susceptible to measurement as **B**—or more so.' To support his conjecture, Davis adopted the *conventional* point of view that in formulating physical laws, we have two kind of quantities: *fundamental* quantities measured in terms of fundamental units and *derived*

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*Author to whom any correspondence should be addressed.

quantities defined in terms of fundamental quantities. Considering that a derived quantity might be a complicated function of fundamental quantities, Davis assumed that is 'reasonable to say that the more complicated this function, the less 'real' it is.' According to this argument, the electric and magnetic fields are less real than the scalar and vector potentials—the electric and magnetic fields depend on the charge and current sources in a more complicated form than the Lorenz-gauge scalar and vector potentials depend on the same sources. On the basis of this conventional argument, Davis suggested to change Feynman's assertion to '**A** is as fundamental as **B**—actually more fundamental.'

To avoid mistakes or misunderstandings about the meaning that Feynman attributed to potentials, we are going to quote Feynman himself. In section 15-4 of volume 2 of *Feynman's Lectures on Physics* [7], Feynman explained why he considered the vector potential to be a **real** field:

> What we mean here by a 'real' field is this: a real field is a mathematical function we use for avoiding the idea of action at a distance … there are phenomena involving quantum mechanics which show that the potential is in fact a 'real' field in the sense we have defined it.

At the end of section 15-5 of the same volume, Feynman also stated that potentials were **fundamental** quantities in quantum mechanics:

> … the vector potential A (together with the scalar potential ϕ that goes with it) appears to give the most direct description of the physics. This becomes more and more apparent the more deeply we go into the quantum theory. In the general theory of quantum electrodynamics, one takes the vector and scalar potentials as the fundamental quantities …

In volume 2, p 45, of the *Feynman Hughes Lectures* [8],[3] we find an interesting section entitled: 'How to detect the vector potential', in which there is a discussion about the existence of the vector potential and how it could be detected using quantum mechanics. A slightly different version of the Aharonov–Bohm experiment [9] is discussed in which the infinite solenoid is replaced by a very large coil in the form of a torus. After a description of the associated AB effect, we read

> For the electrons to be diffracted like this [the shift in the AB effect] they must feel the presence of a new kind of field, viz, the vector potential … it is possible to determine the line integral of 'A' around a closed path even in regions of $B = 0$ … there is a condition in space which we chose to call the vector potential which properly explains the behavior of particles moving in regions of zero B—field … In classical physics we would not be able to directly measure the A—potential's effect as seen by the experiment. One consequence of this argument is that the vector potential cannot be determined at a single point in space rather it is evaluated over a closed path … All the difficulty we have in determining A is due to the fact that we must interfere with the electron to deflect it and measure it …

Considering the context of his paragraph, Feynman seemed to use the word *determining* as a synonym for *measuring*—but this is only our personal opinion. In light of Feynman's previous quotes, we could believe that Feynman would agree with Davis' assertion: '**A** is as

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[3] Between 1966 and 1971 Feynman offered a series of lectures at the Hughes Aircraft Corporation. John T Neer took and transcribed the corresponding notes. They are now freely available in five volumes with the generic name *Feynman Hughes Lectures* [8]. These notes clearly reflect Feynman's teaching style. Nevertheless, we believe that these notes should be taken with some reservation because they were not directly written or endorsed by Feynman himself.

fundamental as **B**—actually more fundamental.' But we do not believe that Feynman would support Davis' assertion: '**A** is as susceptible to measurement as **B**—or more so': again this is our personal opinion. In other words, for Feynman the vector potential was a real but not a directly measurable quantity and was more fundamental than the magnetic field in quantum mechanics. Feynman argued that the potential could be indirectly determined using the AB effect; it is unclear in the last quote, however, if by 'determining' Feynman meant 'measuring'.

2. With regard to our constructive approach, Davis claimed:

> They then use the continuity equation and causality to motivate definitions for these conjectured potentials, saying 'We call these terms the retarded vector potential **A** and the retarded scalar potential Φ.' ('Call' in this context, is equivalent to 'define').

We disagree with Davis. We use the continuity equation and causality to *construct* expressions for two characteristic retarded quantities appearing in our equations and *chose* to call them the retarded vector potential **A** and the retarded scalar potential Φ, but we might as well have chosen other names—what matters is the concept, not the name. In other words, in our context 'to call' is not equivalent 'to define' and therefore the potentials **A** and Φ are to us heuristic constructions based on the physical axiom of charge conservation—mathematically expressed through the local equation of continuity—and the physical requirement of causality—represented either by the retarded time or the retarded Green function of the wave equation. It is worth emphasizing that in constructive approaches one makes use of heuristic arguments to show the *existence* of a mathematical quantity by providing a method for *constructing* it. In our case the constructive approach was an axiomatic-heuristic approach whose application allows us to *construct* the retarded potentials of electrodynamics. But it should be clear that we do not *define* these potentials in the sense that Davis gives to this word, according to which 'when interpreted as definitions, these expressions for the potentials are no longer heuristic.' Our construction of potentials was always heuristic and had an axiom as a starting point.

3. Davis integrated the continuity equation and obtained the expression

$$\rho(\boldsymbol{r}, t) = -\int_{-\infty}^{t} \nabla \cdot \boldsymbol{J}(\boldsymbol{r}, t') \, \mathrm{d}t', \tag{1}$$

where he assumed the condition $\rho(\boldsymbol{r}, -\infty) = 0$. He then claimed

> This procedure defines ρ as a derived quantity in terms of $\boldsymbol{J}$, but can we reverse this procedure to obtain $\boldsymbol{J}$ in terms of ρ? The answer is no.

To support his answer, he invoked the Helmholtz theorem and then concluded that 'We must consider the current density as more fundamental than the charge density.'

Strictly speaking, (1) defines the density ρ in terms of the divergence of the current $\boldsymbol{J}$. Contrary to Davis' claim, there is an expression that defines the current $\boldsymbol{J}$ in terms of the time derivative of density ρ:

$$\boldsymbol{J}(\boldsymbol{r}, t) = \nabla \int \frac{1}{4\pi |\boldsymbol{r} - \boldsymbol{r}'|} \frac{\partial \rho(\boldsymbol{r}', t)}{\partial t} \mathrm{d}^3 r'. \tag{2}$$

Taking the divergence to (2), using the well-known identity $\nabla^2(1/|\boldsymbol{r} - \boldsymbol{r}'|) = -4\pi\delta(\boldsymbol{r} - \boldsymbol{r}')$, where δ is the Dirac delta function and integrating over all space, we recover the continuity equation. In the same sense that Davis says that (1) states that $\boldsymbol{J}$ is more fundamental than ρ, we can say that (2) states that ρ is more fundamental than $\boldsymbol{J}$. Therefore Davis's interpretation that $\boldsymbol{J}$ is more fundamental than ρ is unsatisfactory to say the least. However, it can be

correctly argued that (2) is not an expression of general character and that it only represents the family of irrotational current densities satisfying $\nabla \times \boldsymbol{J}(\boldsymbol{r}, t) = 0$ for each value of $\boldsymbol{r}$ and t. But we can equally argue that (1) is not an expression of general character either and only represents the family of charge densities satisfying the condition $\rho(\boldsymbol{r}, -\infty) = 0$ for each value of $\boldsymbol{r}$. Accordingly, the expressions (1) and (2) have a restricted range of validity and are therefore of limited usefulness in practical applications. We think this is the reason they are not usually mentioned in textbooks. Following the same order of ideas, Davis commented that he showed that[4]

> the vector potential can be logically and compellingly defined in terms only length, time and current density using the retarded Helmholtz theorem, and the scalar potential can be defined in terms of $\boldsymbol{A}$ as

$$\Phi(\boldsymbol{r}, t) = -\int_{-\infty}^{t} \nabla \cdot \boldsymbol{A}(\boldsymbol{r}, t') \, \mathrm{d}t', \tag{3}$$

> which is the time-integrated form of the so-called Lorentz condition.

But the fact is that the potential $\boldsymbol{A}$ can also be *expressed* in terms of the potential Φ:

$$\boldsymbol{A}(\boldsymbol{r}, t) = \nabla \int \frac{1}{4\pi c^2 |\boldsymbol{r} - \boldsymbol{r}'|} \frac{\partial \Phi(\boldsymbol{r}', t)}{\partial t} \mathrm{d}^3 r'. \tag{4}$$

Taking the divergence to (4), using $\nabla^2 (1/|\boldsymbol{r} - \boldsymbol{r}'|) = -4\pi \delta(\boldsymbol{r} - \boldsymbol{r}')$ and integrating over all space, we recover the Lorenz condition: $\nabla \cdot \boldsymbol{A} + (1/c^2)\partial \Phi/\partial t = 0$. Like (1) and (2), and by the same previous argument, (3) and (4) are not of general character and have a limited usefulness in practical applications.

4. Davis pointed out that

> though our development, like the Heras and Heras, also implies that the very concept of gauge invariance is nonvalid within the context of causal electromagnetic theory.

We strongly disagree with this statement. Contrary to what Davis says, gauge invariance is a *valid* symmetry in causal electrodynamics. The crux of Davis's misunderstandings is his insistence that we have *defined* the retarded potentials, and since they are shown to be unique then gauge invariance was thrown out the window. In fact, in our paper [2] we constructed the retarded potentials with their inherent uniqueness, but what we did not say—although we should have said—is that the subsequent heuristic construction of the retarded fields would allow us to introduce an ambiguity by hand and that this would be, *mutatis mutandis*, the gauge invariance. Let us briefly elaborate this last result. We first note that our approach, developed in [2], led us to introduce the following equations for the electric and magnetic fields:

$$\mathbf{E} = -\nabla\Phi - \frac{\partial \mathbf{A}}{\partial t}, \qquad \mathbf{B} = \nabla \times \mathbf{A}, \tag{5}$$

where Φ and $\mathbf{A}$ denote the scalar and vector retarded potentials:

$$\mathbf{A} = \frac{\mu_0}{4\pi} \int \frac{[\mathbf{J}]}{R} \mathrm{d}^3 x', \qquad \Phi = \frac{1}{4\pi\epsilon_0} \int \frac{[\rho]}{R} \mathrm{d}^3 x'. \tag{6}$$

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[4] Here Davis is referring to his paper [10] where he uses a version of the retarded Helmholtz theorem [11, 12] that uses the inverse of the d'Alembertian operator.

Here the square brackets [] denote the retardation symbol indicating that the enclosed quantity is to be evaluated at the source point $\mathbf{x}'$ at the retarded time $t' = t - R/c$ with c being the speed of light, and $R = |\mathbf{x} - \mathbf{x}'|$ is the distance between $\mathbf{x}'$ and the field point $\mathbf{x}$. The volume integrals are taken over all space. We note that the formal structure of the fields $\mathbf{E}$ and $\mathbf{B}$ in (5) allows us to introduce an ambiguity. The left-hand sides of equation (5) remain invariant if we re-write their right-hand sides as

$$\mathbf{E} = -\nabla \left(\Phi - \frac{\partial \Lambda}{\partial t} \right) - \frac{\partial}{\partial t}(\mathbf{A} + \nabla\Lambda), \qquad \mathbf{B} = \nabla \times (\mathbf{A} + \nabla\Lambda), \qquad (7)$$

where $\Lambda = \Lambda(\mathbf{x}, t)$ is an arbitrary (non-singular and single-valued) gauge function, whose insertion was made by hand by adding $\nabla\partial\Lambda/\partial t - \partial\nabla\Lambda/\partial t = 0$ to the right-hand side of the first equation in (5) and $\nabla \times \nabla\Lambda = 0$ to the right-hand side of the second equation in (5). Equation (7) show the *existence* of the primed vector and scalar potentials

$$\mathbf{A}' = \mathbf{A} + \nabla\Lambda, \qquad \Phi' = \Phi - \frac{\partial \Lambda}{\partial t}. \qquad (8)$$

Since Λ is an arbitrary function then the equations in (8) are gauge transformations. We tacitly chose the condition $\Lambda = 0$ in our original approach when arriving at (5). But if we introduce the gauge ambiguity as shown in (7) then we could choose another different condition for Λ.

5. Finally, we find unsatisfactory Davis's argument that the vector potential is more fundamental than the magnetic field. Following his line of argument, the potential $\mathbf{A}$ is more fundamental than field $\mathbf{B}$ because if one *defines* the former then the latter is a *derived* quantity. The problem here is that one cannot clearly discern which is the fundamental quantity and which is the derived one. From the formulas $\mathbf{E} = -\nabla\Phi_{\mathrm{L}} - \partial\mathbf{A}_{\mathrm{L}}/\partial t$ and $\mathbf{B} = \nabla \times \mathbf{A}_{\mathrm{L}}$ it follows that the fields $\mathbf{E}$ and $\mathbf{B}$ are quantities derived from the Lorenz-gauge potentials Φ_{L} and $\mathbf{A}_{\mathrm{L}}$ and, therefore, following Davis's argument, we can claim that the latter are more fundamental than the former. But in reference [14] it has been shown that the Lorenz-gauge potentials can be expressed in terms of the retarded values of the electric and magnetic fields:

$$\Phi_{\mathrm{L}} = -\frac{1}{4\pi} \int \left(\frac{\hat{\mathbf{R}}}{R^2} \cdot [\mathbf{E}] + \frac{\hat{\mathbf{R}}}{Rc} \cdot \left[\frac{\partial \mathbf{E}}{\partial t} \right] \right) \mathrm{d}^3x', \qquad (9)$$

$$\mathbf{A}_{\mathrm{L}} = \frac{1}{4\pi} \int \left([\mathbf{B}] \times \frac{\hat{\mathbf{R}}}{R^2} + \left[\frac{\partial \mathbf{B}}{\partial t} \right] \times \frac{\hat{\mathbf{R}}}{Rc} - \frac{1}{c^2}\frac{1}{R} \left[\frac{\partial \mathbf{E}}{\partial t} \right] \right) \mathrm{d}^3x', \qquad (10)$$

where $\hat{\mathbf{R}} = \mathbf{R}/R = (\mathbf{x} - \mathbf{x}')/|\mathbf{x} - \mathbf{x}'|$. According to these formulas, the potentials Φ_{L} and $\mathbf{A}_{\mathrm{L}}$ are quantities derived from the fields $\mathbf{E}$ and $\mathbf{B}$ and, following Davis's argument, we can equally say that the latter are more fundamental than the former. In this relevant case, Davis's argument does not allow us to identify which set, $\{\mathbf{E}, \mathbf{B}\}$ or $\{\Phi, \mathbf{A}\}$, represents the fundamental quantities.

ORCID iDs

José A Heras ⓘ https://orcid.org/0000-0001-5610-1976
Ricardo Heras ⓘ https://orcid.org/0000-0003-1234-2481

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